

Δ - CONVERGENCE ON TIME SCALE

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ABSTRACT. In the present paper we will give some new notions, such as Δ -convergence and Δ -Cauchy, by using the Δ -density and investigate their relations. It is important to say that, the results presented in this work generalize some of the results mentioned in the theory of statistical convergence.

1. INTRODUCTION AND BACKGROUND

In [1] Fast introduced an extension of the usual concept of sequential limits which he called statistical convergence. In [2] Schoenberg gave some basic properties of statistical convergence. In [3] Fridy introduced the concept of a statistically Cauchy sequence and proved that it is equivalent to statistical convergence.

The theory of time scales was introduced by Hilger in his Ph. D. thesis in 1988, supervised by Auldbach [4]. Measure theory on time scales was first constructed by Guseinov [5], then further studies were made by Cabada-Vivero [6], Rzezuchowski [7]. In [8] Deniz-Ufuktepe define Lebesgue-Stieltjes Δ and ∇ -measures and by using these measures they define an integral adapted to time scale, specifically Lebesgue-Stieltjes Δ -integral.

A time scale \mathbb{T} is an arbitrary nonempty closed subset of the real numbers \mathbb{R} . The time scale \mathbb{T} is a complete metric space with the usual metric. We assume throughout that a time scale \mathbb{T} has the topology that it inherits from the real numbers with the standart topology.

For $t \in \mathbb{T}$ we define the *forward jump operator* $\sigma : \mathbb{T} \rightarrow \mathbb{T}$ by

$$\sigma(t) := \inf \{s \in \mathbb{T} : s > t\}.$$

In this definition we put $\inf\{\emptyset\} = \sup \mathbb{T}$.

For $a, b \in \mathbb{T}$ with $a \leq b$ we define the interval $[a, b]$ in \mathbb{T} by

$$[a, b] = \{t \in \mathbb{T} : a \leq t \leq b\}.$$

Open intervals and half-open intervals etc. are defined accordingly.

Let \mathbb{T} be time scale. Denote by \mathcal{S} the family of all left-closed and right-open intervals of \mathbb{T} of the form $[a, b) = \{t \in \mathbb{T} : a \leq t < b\}$ with $a, b \in \mathbb{T}$ and $a \leq b$. The interval $[a, a)$ is understood as the empty set. \mathcal{S} is a semiring of subsets of \mathbb{T} . Obviously the set function $m : \mathcal{S} \rightarrow [0, \infty]$ defined by $m([a, b)) = b - a$ is a countably additive measure. An outer measure $m^* : \mathcal{P}(\mathbb{T}) \rightarrow [0, \infty]$ generated by m is defined by

$$m^*(A) = \inf \left\{ \sum_{n=1}^{\infty} m(A_n) : (A_n) \text{ is a sequence of } \mathcal{S} \text{ with } A \subset \bigcup_{n=1}^{\infty} A_n \right\}.$$

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If there is no sequence (A_n) of \mathcal{S} such that $A \subset \bigcup_{n=1}^{\infty} A_n$, then we let $m^*(A) = \infty$. We define the family $\mathcal{M}(m^*)$ of all m^* -measurable subsets of \mathbb{T} , i.e.

$$\mathcal{M}(m^*) = \{E \subset \mathbb{T} : m^*(A) = m^*(A \cap E) + m^*(A \cap E^c) \text{ for all } A \subset \mathbb{T}\}.$$

The collection $\mathcal{M}(m^*)$ of all m^* -measurable sets is a σ -algebra and the restriction of m^* to $\mathcal{M}(m^*)$ which we denote by μ_{Δ} is a countably additive measure on $\mathcal{M}(m^*)$. This measure μ_{Δ} which is Carathéodory extension of the set function m associated with the family \mathcal{S} we call Lebesgue Δ -measure on \mathbb{T} .

We call $f : \mathbb{T} \rightarrow \mathbb{R}$ is a measurable function, if $f^{-1}(\mathcal{O}) \in \mathcal{M}(m^*)$ for every open subset of \mathcal{O} of \mathbb{R} .

Theorem 1. (see [5]) *For each t_0 in $\mathbb{T} - \{\max \mathbb{T}\}$ the single point set $\{t_0\}$ is Δ -measurable and its Δ -measure is given by*

$$\mu_{\Delta}(\{t_0\}) = \sigma(t_0) - t_0.$$

Theorem 2. (see [5]) *If $a, b \in \mathbb{T}$ and $a \leq b$, then*

$$\mu_{\Delta}([a, b)) = b - a, \quad \mu_{\Delta}((a, b)) = b - \sigma(a).$$

If $a, b \in \mathbb{T} - \{\max \mathbb{T}\}$ and $a \leq b$, then

$$\mu_{\Delta}((a, b]) = \sigma(b) - \sigma(a), \quad \mu_{\Delta}([a, b]) = \sigma(b) - a.$$

It can easily seen from Theorem 1 that, the measure of a subset of \mathbb{N} is equal to its cardinality.

2. Δ -DENSITY, Δ -CONVERGENCE, Δ -CAUCHY

It is well-known that the notions of statistical convergence and statistical Cauchy are closely related with the density of the subset of \mathbb{N} . In the present section first of all we will define the density of the subset of the time scale. By using this definition, we will define Δ -convergence and Δ -Cauchy for a real valued function defined on the time scale. After then we will show that these notions are equivalent.

Throughout this paper we consider the time scales, which are unbounded from above and has a minimum point.

Let A be a Δ -measurable subset of \mathbb{T} and let $a = \min \mathbb{T}$. Δ -density of A in \mathbb{T} (or briefly Δ -density of A) is defined by

$$\delta_{\Delta}(A) = \lim_{t \rightarrow \infty} \frac{\mu_{\Delta}(A(t))}{\sigma(t) - a}$$

(if this limit exists) where

$$(2.1) \quad A(t) = \{s \in A : s \leq t\}$$

and $t \in \mathbb{T}$.

From the identity $A(t) = A \cap [a, t]$, measurability of A implies measurability of $A(t)$.

If $f : \mathbb{T} \rightarrow \mathbb{R}$ is a function such that $f(t)$ satisfies property P for all t except a set of Δ -density zero, then we say that $f(t)$ satisfies P for " Δ -almost all t ", and we abbreviate this by " Δ -a.a. t ". Let

$$\begin{aligned} \mathcal{M}_d & : = \{A \in \mathbb{T} : \Delta\text{-density of } A \text{ is exists in } \mathbb{T}\}, \\ \mathcal{M}_d^0 & : = \{A \in \mathbb{T} : \delta_{\Delta}(A) = 0\}. \end{aligned}$$

Lemma 1. i) If $A, B \in \mathcal{M}_d$ and $A \subset B$, then $\delta_\Delta(A) \leq \delta_\Delta(B)$,
 ii) If $A \in \mathcal{M}_d$, then $0 \leq \delta_\Delta(A) \leq 1$,
 iii) $\mathbb{T} \in \mathcal{M}_d$ and $\delta_\Delta(\mathbb{T}) = 1$,
 iv) If $A \in \mathcal{M}_d$, then $A^c \in \mathcal{M}_d$ and $\delta_\Delta(A) + \delta_\Delta(A^c) = 1$,
 v) If $A, B \in \mathcal{M}_d$ and $A \subset B$, then $B - A \in \mathcal{M}_d$ and $\delta_\Delta(B - A) = \delta_\Delta(B) - \delta_\Delta(A)$,
 vi) If A_1, A_2, \dots, A_n is a mutually disjoint sequence in \mathcal{M}_d , then $\bigcup_{k=1}^n A_k \in \mathcal{M}_d$
 and

$$\delta_\Delta \left(\bigcup_{k=1}^n A_k \right) = \sum_{k=1}^n \delta_\Delta(A_k),$$

vii) If $A_1, A_2, \dots, A_n \in \mathcal{M}_d$ and $\bigcup_{k=1}^n A_k \in \mathcal{M}_d$, then

$$\delta_\Delta \left(\bigcup_{k=1}^n A_k \right) \leq \sum_{k=1}^n \delta_\Delta(A_k),$$

viii) If A is measurable set and $B \in \mathcal{M}_d^0$ with $A \subset B$ then $A \in \mathcal{M}_d^0$,
 ix) If $A_1, A_2, \dots, A_n \in \mathcal{M}_d^0$, then $\bigcup_{k=1}^n A_k, \bigcap_{k=1}^n A_k \in \mathcal{M}_d^0$,
 x) Every bounded measurable subset of \mathbb{T} belongs to \mathcal{M}_d^0 ,
 xi) If $A \in \mathcal{M}_d^0$ and $B \in \mathcal{M}_d$ then $\delta_\Delta(A \cup B) = \delta_\Delta(B)$.

Proof. i) Let $A, B \in \mathcal{M}_d$ and $A \subset B$. Clearly $A(t) \subset B(t)$ and since μ_Δ is a measure function, one has $\mu_\Delta(A) \leq \mu_\Delta(B)$. Thus we have;

$$\delta_\Delta(A) = \lim_{t \rightarrow \infty} \frac{\mu_\Delta(A(t))}{\sigma(t) - a} \leq \lim_{t \rightarrow \infty} \frac{\mu_\Delta(B(t))}{\sigma(t) - a} = \delta_\Delta(B)$$

ii) Note that $A(t) \subset [a, t]$. The required inequalities follows from the following inequalities;

$$0 \leq \frac{\mu_\Delta(A(t))}{\sigma(t) - a} \leq \frac{\mu_\Delta([a, t])}{\sigma(t) - a} = \frac{\sigma(t) - a}{\sigma(t) - a} = 1.$$

iii) It is clear that \mathbb{T} is measurable. The Δ -density of \mathbb{T} is obtained from the following equalities;

$$\delta_\Delta(\mathbb{T}) = \lim_{t \rightarrow \infty} \frac{\mu_\Delta(\mathbb{T}(t))}{\sigma(t) - a} = \lim_{t \rightarrow \infty} \frac{\mu_\Delta([a, t])}{\sigma(t) - a} = 1$$

iv) Since A is measurable so is A^c , namely $A^c(t) = \{s \leq t : s \in A^c\}$ is measurable. On the other hand $A(t) \cup A^c(t) = [a, t]$ and

$$1 = \frac{\mu_\Delta([a, t])}{\sigma(t) - a} = \frac{\mu_\Delta(A(t))}{\sigma(t) - a} + \frac{\mu_\Delta(A^c(t))}{\sigma(t) - a}$$

imply that the required statement holds as $t \rightarrow \infty$.

v) Since A and B are measurable so are $B - A$. The statement can be easily shown by considering $A(t) \cup (B - A)(t) \cup B^c(t) = [a, t]$.

vi) Since the Δ -density of for each subset A_k exists, one can write

$$\begin{aligned}
 \sum_{k=1}^n \delta_{\Delta}(A_k) &= \sum_{k=1}^n \lim_{t \rightarrow \infty} \frac{\mu_{\Delta}(A_k(t))}{\sigma(t) - a} \\
 &= \lim_{t \rightarrow \infty} \sum_{k=1}^n \frac{\mu_{\Delta}(A_k(t))}{\sigma(t) - a} \\
 &= \lim_{t \rightarrow \infty} \frac{\mu_{\Delta}\left(\bigcup_{k=1}^n A_k(t)\right)}{\sigma(t) - a} \\
 &= \lim_{t \rightarrow \infty} \frac{\mu_{\Delta}\left(\left(\bigcup_{k=1}^n A_k\right)(t)\right)}{\sigma(t) - a} \\
 &= \delta_{\Delta}\left(\bigcup_{k=1}^n A_k\right)
 \end{aligned}$$

vii) The proof is similar to that of the previous proof.

viii) It can be easily seen from (i).

ix) Considering $\delta_{\Delta}(A_k) = 0$ for $k = 1, 2, \dots, n$ and (vii), one can obtain $\bigcup_{k=1}^n A_k \in \mathcal{M}_d^0$. And $\bigcap_{k=1}^n A_k \in \mathcal{M}_d^0$ can be obtain from (viii).

x) Let A be a bounded set. For a sufficiently large $M \in \mathbb{T}$ we can write $A \subset [a, M]$. Then one has

$$0 \leq \frac{\mu_{\Delta}(A(t))}{\sigma(t) - a} \leq \frac{\mu_{\Delta}([a, M])}{\sigma(t) - a} \rightarrow 0 \quad (t \rightarrow \infty),$$

this implies that $\delta_{\Delta}(A) = 0$.

xi) (i) and (vii) yields $\delta_{\Delta}(B) \leq \delta_{\Delta}(A \cup B) \leq \delta_{\Delta}(A) + \delta_{\Delta}(B) = \delta_{\Delta}(B)$. This completes the proof. \square

It is clear that the familiy \mathcal{M}_d^0 is a ring of subsets of \mathbb{T} . According to (iv), the Δ -density of the complement of a subset whose Δ -density is 0 is equal to 1, \mathcal{M}_d^0 is not closed under the operation complement. So it is not an algebra. Note that the Δ -density of a subset of \mathbb{N} is equal to its natural density.

Example 1. Let $\mathbb{T} = [0, \infty)$, l and r be arbitrary two positive real numbers. Let also $A = \bigcup_{n=0}^{\infty} A_n$, where $A_n = [nl + nr, (n+1)l + nr]$. According to Lemma 1-(x) for each A_n is bounded and so $\delta_{\Delta}(A_n) = 0$. In addition, let $A(t)$ defined as in (2.1), we have

$$\mu_{\Delta}(A(t)) = \begin{cases} t - nr & , \quad nl + nr \leq t \leq (n+1)l + nr \\ (n+1)l & , \quad (n+1)l + nr \leq t \leq (n+1)l + (n+1)r \end{cases}$$

($n = 0, 1, 2, \dots$) and hence

$$\delta_{\Delta}(A) = \lim_{t \rightarrow \infty} \frac{\mu_{\Delta}(A(t))}{\sigma(t) - a} = \lim_{t \rightarrow \infty} \frac{\mu_{\Delta}(A(t))}{t - 0} = \frac{l}{l + r}.$$

Note that, since

$$\frac{l}{l + r} = \delta_{\Delta}(A) > \sum_{n=0}^{\infty} \delta_{\Delta}(A_n) = 0$$

δ_{Δ} does not define a measure.

Example 2. Let $\mathbb{T} = \mathbb{N}$. The Δ -density of $A = \{2, 4, 6, \dots\}$ in \mathbb{N} is given by

$$\delta_{\Delta}(A) = \lim_{t \rightarrow \infty} \frac{\mu_{\Delta}(A(t))}{\sigma(t) - a} = \lim_{t \rightarrow \infty} \frac{[t/2]}{t} = \frac{1}{2}.$$

Definition 1. (Δ -convergence) The function $f : \mathbb{T} \rightarrow \mathbb{R}$ is Δ -convergent to the number L provided that for each $\varepsilon > 0$, there exists $K_{\varepsilon} \subset \mathbb{T}$ such that $\delta_{\Delta}(K_{\varepsilon}) = 1$ and $|f(t) - L| < \varepsilon$ holds for all $t \in K_{\varepsilon}$.

We will use notation $\Delta\text{-}\lim_{t \rightarrow \infty} f(t) = L$.

Definition 2. (Δ -Cauchy) The function $f : \mathbb{T} \rightarrow \mathbb{R}$ is Δ -Cauchy provided that for each $\varepsilon > 0$, there exists $K_{\varepsilon} \subset \mathbb{T}$ and $t_0 \in \mathbb{T}$ such that $\delta_{\Delta}(K_{\varepsilon}) = 1$ and $|f(t) - f(t_0)| < \varepsilon$ holds for all $t \in K_{\varepsilon}$.

Proposition 1. Let $f : \mathbb{T} \rightarrow \mathbb{R}$ be a measurable function. $\Delta\text{-}\lim_{t \rightarrow \infty} f(t) = L$ if and only if for each $\varepsilon > 0$, $\delta_{\Delta}(\{t \in \mathbb{T} : |f(t) - L| \geq \varepsilon\}) = 0$.

Proof. Let $\Delta\text{-}\lim_{t \rightarrow \infty} f(t) = L$ and $\varepsilon > 0$ be given. In this case there exists a subset $K_{\varepsilon} \subset \mathbb{T}$ such that $\delta_{\Delta}(K_{\varepsilon}) = 1$ and $|f(t) - L| < \varepsilon$ holds for all $t \in K_{\varepsilon}$. Since $K_{\varepsilon} \subset \{t \in \mathbb{T} : |f(t) - L| < \varepsilon\}$ we obtain $\delta_{\Delta}(\{t \in \mathbb{T} : |f(t) - L| < \varepsilon\}) = 1$. Hence we get $\delta_{\Delta}(\{t \in \mathbb{T} : |f(t) - L| \geq \varepsilon\}) = 0$.

Other case of the proof is straightforward. \square

Proposition 2. Let $f : \mathbb{T} \rightarrow \mathbb{R}$ be a measurable function. f is Δ -Cauchy if and only if for each $\varepsilon > 0$, there exists $t_0 \in \mathbb{T}$ such that $\delta_{\Delta}(\{t \in \mathbb{T} : |f(t) - f(t_0)| \geq \varepsilon\}) = 0$.

Example 3. Let $\mathbb{I}_{[0, \infty)}$ be irrational numbers and $\mathbb{Q}_{[0, \infty)}$ be rational numbers in $[0, \infty)$. Let consider the function $f : \mathbb{T} = [0, \infty) \rightarrow \mathbb{R}$ defined as following,

$$f(t) = \begin{cases} 0 & t \in \mathbb{Q}_{[0, \infty)} \\ 1 & t \in \mathbb{I}_{[0, \infty)} \end{cases}.$$

Since $\mu_{\Delta}(\mathbb{Q}_{[0, \infty)}) = 0$, the density of the subset $\mathbb{Q}_{[0, \infty)}$ in the time scale \mathbb{T} is zero. This implies that $\delta_{\Delta}(\mathbb{I}_{[0, \infty)}) = 1$. So, for each $\varepsilon > 0$ and for all $t \in \mathbb{I}_{[0, \infty)}$ one has $0 = |f(t) - 1| < \varepsilon$ and as a corollary, we get $\Delta\text{-}\lim_{t \rightarrow \infty} f(t) = 1$.

Proposition 3. The Δ -limit of a function $f : \mathbb{T} \rightarrow \mathbb{R}$ is unique.

Proof. Let $\Delta\text{-}\lim_{t \rightarrow \infty} f(t) = L_1$ and $\Delta\text{-}\lim_{t \rightarrow \infty} f(t) = L_2$. Let $\varepsilon > 0$ be given. Then there exist subsets $K_1, K_2 \subset \mathbb{T}$ such that for every $t \in K_1$ with $|f(t) - L_1| < \varepsilon/2$ and for every $t \in K_2$ with $|f(t) - L_1| < \varepsilon/2$, where $\delta_{\Delta}(K_1) = 1$ and $\delta_{\Delta}(K_2) = 1$. From Lemma 1-(iv) we have $K_1 \cap K_2 \neq \emptyset$. Thus for every $t \in K_1 \cap K_2$ one has

$$|L_1 - L_2| \leq |f(t) - L_1| + |f(t) - L_2| < \varepsilon.$$

Thus $L_1 = L_2$. \square

Proposition 4. If $f, g : \mathbb{T} \rightarrow \mathbb{R}$ with $\Delta\text{-}\lim_{t \rightarrow \infty} f(t) = L_1$ and $\Delta\text{-}\lim_{t \rightarrow \infty} g(t) = L_2$, then the following statements hold:

- i) $\Delta\text{-}\lim_{t \rightarrow \infty} (f(t) + g(t)) = L_1 + L_2$,
- ii) $\Delta\text{-}\lim_{t \rightarrow \infty} (cf(t)) = cL_1, (c \in \mathbb{R})$.

Proposition 5. If $f : \mathbb{T} \rightarrow \mathbb{R}$ with $\lim_{t \rightarrow \infty} f(t) = L$ then $\Delta\text{-}\lim_{t \rightarrow \infty} f(t) = L$.

Proof. Let $\lim_{t \rightarrow \infty} f(t) = L$. In this case for a given $\varepsilon > 0$ we can find a $t_0 \in \mathbb{T}$ such that $|f(t) - L| < \varepsilon$ holds for every $t > t_0$. The set $K_\varepsilon = \{t \in \mathbb{T} : t > t_0\}$ is measurable and from Lemma 1 (iv) and (x) one has $\delta_\Delta(K_\varepsilon) = 1$. By the definition of Δ -convergence we get $\Delta\text{-}\lim_{t \rightarrow \infty} f(t) = L$. \square

Theorem 3. *Let $f : \mathbb{T} \rightarrow \mathbb{R}$ be a measurable function. The following statements are equivalent:*

- i) f is Δ -convergent,
- ii) f is Δ -Cauchy,
- iii) *There exists a measurable and convergent function $g : \mathbb{T} \rightarrow \mathbb{R}$ such that $f(t) = g(t)$ for Δ -a.a. t .*

Proof. (i) \Rightarrow (ii) : Let $\Delta\text{-}\lim_{t \rightarrow \infty} f(t) = L$ and $\varepsilon > 0$ be given. Then $|f(t) - L| < \varepsilon/2$ holds for Δ -a.a. t . We can choose $t_0 \in \mathbb{T}$ such that $|f(t_0) - L| < \varepsilon/2$ holds. So,

$$|f(t) - f(t_0)| \leq |f(t) - L| + |f(t_0) - L| < \varepsilon \text{ } \Delta\text{-a.a. } t.$$

This shows that f satisfies the property of Δ -Cauchy.

(ii) \Rightarrow (iii) : We can choose an element $t_1 \in \mathbb{T}$. We can define an interval $I = [f(t_1) - 1, f(t_1) + 1]$ which contains $f(t)$ for Δ -a.a. t . By the same method we can choose an element $t^* \in \mathbb{T}$ and define an interval $I' = [f(t^*) - 1/2, f(t^*) + 1/2]$ which contains $f(t)$ for Δ -a.a. t . We can write

$$\{t \leq s : f(t) \notin I \cap I'\} = \{t \leq s : f(t) \notin I\} \cup \{t \leq s : f(t) \notin I'\}.$$

Since f is a measurable function, the two terms appear on the righthand-side of the last equality are also measurable. By using Lemma 1-(vii) we obtain

$$\begin{aligned} \delta_\Delta(\{t \in \mathbb{T} : f(t) \notin I \cap I'\}) &\leq \delta_\Delta(\{t \in \mathbb{T} : f(t) \notin I\}) + \delta_\Delta(\{t \in \mathbb{T} : f(t) \notin I'\}) \\ &= 0. \end{aligned}$$

So, $f(t)$ is in the closed interval $I_1 = I \cap I'$ for Δ -a.a. t . It is clear that the length of the interval I_1 is less than or equal to 1. Now we can choose $t_2 \in \mathbb{T}$ with $I'' = [f(t_2) - 1/4, f(t_2) + 1/4]$ contains $f(t)$ for Δ -a.a. t . Undoubtly the closed interval $I_2 = I_1 \cap I''$ contains $f(t)$ for Δ -a.a. t and the length of the interval I_2 is less than or equal to 1/2. With the same procedure, for each m we can obtain a sequence of closed intervals $(I_m)_{m=1}^\infty$ such that $I_{m+1} \subset I_m$ and the length of each interval I_m is less than or equal to 2^{1-m} . Moreover $f(t) \in I_m$ for Δ -a.a. t . From the properties of the intersection of closed intervals there exists a real number λ such that $\bigcap_{m=1}^\infty I_m = \{\lambda\}$. Since the Δ -density of the set on which $f(t) \notin I_m$ is equal to zero, we can find an increasing sequence $(T_m)_{m=1}^\infty$ in \mathbb{T} such that,

$$(2.2) \quad \frac{\mu_\Delta(\{t \leq s : f(t) \notin I_m\})}{\sigma(s) - a} < \frac{1}{m}, \quad s > T_m.$$

Here $a = \min \mathbb{T}$. Let consider the function $g : \mathbb{T} \rightarrow \mathbb{R}$ defined as following:

$$g(t) = \begin{cases} \lambda & , T_m < t \leq T_{m+1} \text{ and } f(t) \notin I_m \\ f(t) & , \text{ otherwise.} \end{cases}$$

It is clear that g is a measurable function and $\lim_{t \rightarrow \infty} g(t) = \lambda$. Indeed, for $t > T_m$ either $g(t) = \lambda$ or $g(t) = f(t) \in I_m$. In this case $|g(t) - \lambda| \leq 2^{1-m}$ holds.

Finally we shall show that $f(t) = g(t)$ for Δ -a.a. t . For this purpose consider,

$$\{t \leq s : f(t) \neq g(t)\} \subset \{t \leq s : f(t) \notin I_m\},$$

$T_m < s \leq T_{m+1}$. Thus we get from (2.2)

$$\frac{\mu_{\Delta}(\{t \leq s : f(t) \neq g(t)\})}{\sigma(s) - a} \leq \frac{\mu_{\Delta}(\{t \leq s : f(t) \notin I_m\})}{\sigma(s) - a} < \frac{1}{m}$$

this yielding $\delta_{\Delta}(\{t \in \mathbb{T} : f(t) \neq g(t)\}) = 0$, that is, $f(t) = g(t)$ for Δ -a.a. t .

(iii) \Rightarrow (i) : Let $f(t) = g(t)$ for Δ -a.a. t . and $\lim_{t \rightarrow \infty} g(t) = L$. For a given $\varepsilon > 0$ we have,

$$\{t \in \mathbb{T} : |f(t) - L| \geq \varepsilon\} \subset \{t \in \mathbb{T} : f(t) \neq g(t)\} \cup \{t \in \mathbb{T} : |g(t) - L| \geq \varepsilon\}.$$

Since $\lim_{t \rightarrow \infty} g(t) = L$, the second set appears on the righthand-side of the above inclusion relation is bounded and thus $\delta_{\Delta}(\{t \in \mathbb{T} : |g(t) - L| \geq \varepsilon\}) = 0$. In addition $f(t) = g(t)$ for Δ -a.a. t yields $\delta_{\Delta}(\{t \in \mathbb{T} : f(t) \neq g(t)\}) = 0$. In conclusion $\delta_{\Delta}(\{t \in \mathbb{T} : |f(t) - L| \geq \varepsilon\}) = 0$, namely Δ - $\lim_{t \rightarrow \infty} f(t) = L$. \square

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